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HEAT TRANSFER AT HIGH PRANDTL NUMBERS AND MASS TRANSFER TO A TURBULENT FLOW FROM THE WALL OF A SMOOTH TUBE

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Аннотация—Приводится вывод уравнения тепло- и массоотдачи при $Pr \gg 1$, основанный на рассмотрении переноса в одном лишь вязком подслое. Принимается, что коэффициент турбулентного переноса в вязком подслое пропорционален квадрату расстояния от стенки. Результаты рассчета сопоставляются с опытнымн даннымн.

NOMENCLATURE

- l, mixing length;
- y, distance from the wall;
- $\varkappa$, proportionality factor relating the mixing length with the distance from the wall in the viscous sublayer;
- ω_y', transverse pulsation velocity;
- v_o, dynamic velocity;
- ν, kinematic viscosity;
- D_T, turbulent diffusivity;
- r, distance from the pipe axis;
- C, concentration of the diffusing material;
- ω, averaged velocity at the distance r from the axis;
- x, longitudinal co-ordinate;
- Re^*, $= (v_o R)/(\nu)$;
- η, dimensionless transverse co-ordinate;
- R, pipe radius;
- C', dimensionless concentration;
- ξ, dimensionless longitudinal co-ordinate;
- C_{sat}, saturation concentration;
- C_{ent}, concentration at the entrance to the pipe;
- C_{out}, concentration at the outlet of the pipe;
- ξ_L, dimensionless pipe length;
- $\bar{\omega}$, mean cross-sectional flow velocity;
- α, mass-transfer coefficient;
- L, pipe length;
- ζ, resistance of the pipe;
- Nu_ξ, local Nusselt number;
- Nu_{av}, average Nusselt number;
- D, molecular diffusivity.

A LARGE NUMBER OF DATA on the kinetics of the solution of solids show that in mass transfer the Nusselt number is proportional to the 0·5th power of the Prandtl number [1–3]. On the other hand, the theoretical analysis of the process at $Pr \gg 1$ shows that the exponent in the relation

$$Nu \sim Pr^n$$

depends on the law adopted for the attenuation of the turbulent transfer coefficient in the viscous sublayer. When $n = 0{\cdot}5$, the turbulent transfer coefficient should be proportional to the squared distance from the solid wall. Such a law may be obtained, if Prandtl's concepts are extended through the viscous sublayer up to the wall, i.e. if it is assumed that the mixing length in the viscous sublayer is also proportional to the distance from the wall

$$l = \varkappa y \tag{1}$$

and the transverse component of the pulsation velocity is equal to the change of the mean velocity over the mixing length

$$\omega_y' = \frac{\varkappa v_0^2 y}{\nu}. \tag{2}$$

With these assumptions, the turbulent transfer coefficient of heat or mass in the viscous sublayer is defined by the expression

$$D_T = \frac{\varkappa^2 v_0^2 y^2}{\nu}. \tag{3}$$

At $Pr \gg 1$ we may limit our considerations to the transfer in the viscous sublayer only. This comes from the fact that, due to the considerable excess of the turbulent transfer over the molecular one in the upper layers of the viscous sublayer, the concentration of the diffusing material towards the boundary of the viscous sublayer approaches asymptotically a limit value which solely depends on the longitudinal co-ordinate. This allows us, when defining the boundary conditions of the problem, to neglect the transition layer $[\eta = (v_0 y/\nu) > 6]$ and to assume that the concentration or temperature at $\eta = 6$ is a function of the longitudinal co-ordinate solely.

In the analysis that follows mass transfer is considered, but exactly similar arguments would be used for a heat-transfer problem.

We shall consider a stationary axi-symmetrical flow and assume that radial diffusion exceeds considerably the axial one. For the case, the differential mass-transfer equation is

$$\frac{\partial}{\partial r}\left[r(D + D_r)\frac{\partial C}{\partial r}\right] = \omega r \frac{\partial C}{\partial x} \tag{4}$$

Substituting (3) into (4) and changing to dimensionless values, we get equation (4) in the form:

$$Re^* \frac{\partial}{\partial \eta}\left[(Re^* - \eta)(Pr^{-1} + \varkappa^2\eta^2)\frac{\partial C'}{\partial \eta}\right] = (Re^*\eta - \eta^2)\frac{\partial C'}{\partial \xi}. \tag{5}$$

Where

$$\eta = \frac{v_0 y}{\nu} = \frac{v_0(R - r)}{\nu}; \quad Re^* = \frac{v_0 R}{\nu};$$

$$C' = \frac{C_{sat} - C}{\overline{\Delta C}}; \quad \xi = \frac{x}{R};$$

$$\overline{\Delta C} = \frac{C_{out} - C_{ent}}{\ln \dfrac{C_{sat} - C_{ent}}{C_{sat} - C_{out}}}. \tag{6}$$

Since only the region of the viscous sublayer is considered, it is obvious that $Re^* \gg \eta$ and equation (5) may be simplified to:

$$\frac{\partial}{\partial \eta}\left[\left(\frac{Re^*}{Pr} + \varkappa^2\, Re^*\, \eta^2\right)\frac{\partial C'}{\partial \eta}\right] = \eta \frac{\partial C'}{\partial \xi}. \tag{7}$$

Equation (7) should be solved with the following boundary conditions: $C' = 0$ at the solid wall, i.e. at $\eta = 0$; and $C' = \phi(\xi)$ at $\eta = 6$, i.e. at the external boundary of the viscous sublayer. The boundary conditions along the co-ordinate ξ will be considered below. We shall assume the solution of equation (7) to be of the form

$$C' = af(\eta)\exp(b\xi) \tag{8}$$

(where a and b are constants). This reduces the problem to the solution of an ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\left(\frac{Re^*}{Pr} + \varkappa^2 Re^* \eta^2\right)\frac{\mathrm{d}f}{\mathrm{d}\eta}\right] = b\eta f, \tag{9}$$

in which

$$f = 0 \quad \text{at } \eta = 0 \quad \text{and} \quad f = 1 \quad \text{at } \eta = 6 \tag{10}$$

An approximate solution of equation (9) may be found if its right-hand side is neglected. The solution is of the form

$$f = \frac{\operatorname{arc\,tg}(\varkappa Pr^{0.5}\eta)}{\operatorname{arc\,tg}(6\varkappa Pr^{0.5})} \tag{11}$$

However, a more exact solution is readily obtained when (11) is substituted into the right-hand side of equation (9). Then using the boundary conditions (10), we get:

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)_{\eta=0} = \frac{1}{Re^*}\left\{\frac{Pr^{0.5}(\varkappa^2 Re^* - 3b)}{\varkappa \operatorname{arc\,tg}(b\varkappa Pr^{0.5})} + \frac{b \ln(1 - 36\varkappa^2 Pr)}{2\varkappa^2[\operatorname{arc\,tg}(6\varkappa Pr^{0.5})]^2}\right\}. \tag{12}$$

Again, using relation (8) and remembering that

$$C' = \frac{C_{sat} - C}{\overline{\Delta C}}, \quad \eta = \frac{v_0 y}{\nu} \quad \text{and} \quad Re^* = \frac{v_0 R}{\nu}$$

we obtain

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)_{\eta=0} = -\frac{\nu \exp(-b\xi)}{a\overline{\Delta C} v_0}\left(\frac{\partial C}{\partial y}\right)_{y=0},$$

and from that the local Nusselt number is:

$$-\frac{2R}{\overline{\Delta C}\cdot D}\left(\frac{\partial C}{\partial y}\right)_{y=0} D = Nu_\xi = 2a\exp(b\xi)\left\{\frac{Pr^{0.5}(\varkappa^2 Re^* - 3b)}{\varkappa \operatorname{arc\,tg}(b\varkappa Pr^{0.5})} + \frac{b\ln(1+36\varkappa^2 Pr)}{2\varkappa^2[\operatorname{arc\,tg}(b\varkappa Pr^{0.5})]^2}\right\}. \quad (13)$$

For the mean Nusselt number (over the tube length), the expression is:

$$Nu_{av} = \frac{2a}{b\xi_L}[\exp(b\xi_L) - 1]\left\{\frac{Pr^{0.5}(\varkappa^2 Re^* - 3b)}{\varkappa \operatorname{arc\,tg}(6\varkappa Pr^{0.5})} + \frac{b\ln(1+36\varkappa^2 Pr)}{2\varkappa^2[\operatorname{arc\,tg}(6\varkappa Pr^{0.5})]^2}\right\}. \quad (14)$$

The average Nu should be independent of the pipe length ξ_L in a fully developed flow. This may easily be proved.

We have assumed above that concentration depends on the distance from the wall only within the viscous sublayer. Outside this very thin layer, concentration is a function of a longitudinal co-ordinate only

$$C' = \frac{C_{sat} - C}{\overline{\Delta C}} = a\exp(b\xi). \quad (15)$$

From this at $\xi = 0$ and $\xi = \xi_L$ we have respectively

$$\left.\begin{aligned}&\frac{C_{sat} - C_{ent}}{\overline{\Delta C}} = a\\ &\text{and}\\ &C_{sat} - C_{out} = (C_{sat} - C_{ent})\exp(b\xi_L).\end{aligned}\right\} \quad (16)$$

Now let us make use of relations (6) and

$$R\bar{\omega}(C_{out} - C_{ent}) = 2La\,\overline{\Delta C}. \quad (17)$$

From equations (6), (16) and (17) it is found that

$$a = -\frac{b\bar{\omega}}{2} \quad (18)$$

$$a = \frac{b\xi_L}{\exp(b\xi_L) - 1}. \quad (19)$$

The dimensionless form of (18) is

$$Nu_{av} = -\frac{b}{2}Pe \quad (20)$$

Substitution of (19) into (14) yields

$$Nu_{ev} = 2\left\{\frac{Pr^{0.5}(\varkappa^2 Re^* - 3b)}{\varkappa \operatorname{arc\,tg}(6\varkappa Pr^{0.5})} + \frac{b\ln(1+36\varkappa^2 Pr)}{2\varkappa^2[\operatorname{arc\,tg}(6\varkappa Pr^{0.5})]^2}\right\} \quad (21)$$

Using (20) and the relation

$$Re^* = \frac{Rv_0}{\nu} = Re\left(\frac{\zeta}{32}\right)^{0.5}$$

we get the following formula for the calculation of the average Nusselt number

$$Nu_{av} = \frac{\varkappa\left(\frac{\zeta}{8}\right)^{0.5} Pr^{0.5}\,Re}{\operatorname{arc\,tg}(6\varkappa Pr^{0.5}) - \frac{12Pr^{0.5}}{\varkappa Pe} + \frac{\ln(1+36\varkappa^2 Pr)}{\varkappa^2 Pe\operatorname{arc\,tg}(6\varkappa Pr^{0.5})}} \quad (22)$$

The analysis of the available experimental data on heat transfer at high Prandtl numbers, and of the author's experimental data on mass transfer from the pipe wall to a turbulent flow, shows that $\varkappa \approx 0{\cdot}050$. At $Pr > 10^6$, the second and third terms in the denominator of equation (22) may be neglected, so that the following predicted formula is obtained

$$Nu = \frac{0{\cdot}0177\,\zeta^{0.5} Pr^{0.5}\,Re}{\operatorname{arc\,tg}(0{\cdot}30\,Pr^{0.5})} \quad (23)$$

It should be emphasized that equation (23) leads to the relation $Nu \sim Pr^{0.5}$ at very large Prandtl numbers only, when the denominator of the equation is fairly insensitive to the changes of this number. With decreasing Pr, the effect of the denominator increases which is equivalent to a decrease of the exponent of Pr.

Equation (23) may also be obtained directly from an approximated solution of equation (11).

It is of interest to compare the Nusselt numbers predicted by equation (23) with the data available in the literature.

Linton and Sherwood [4] studied the kinetics of the solution in water of the internal surfaces of small hollow cylinders made of β-naphthol, benzoic and cinnamic acids, axially connected to

the end of a pipe of the same internal diameter. The rate of the process was measured according to the changes of the cylinder weight during the test. Unfortunately, the experimental data of Linton and Sherwood for the turbulent region are probably not reliable, possibly because of the method of measuring the rate of solution adopted by the authors. Their data for pipes of 1·9 cm diameter for benzoic acid give Nusselt numbers which are three times larger than those for β-naphthol and cinnamic acid. The authors suggested that this difference might be attributed to the small cracks in the benzoic acid cylinders found after the experiments, and they considered the results on cinnamic acid and β-naphthol as more reliable. The latter data were also used later by Deissler [5] for his correlation. At the same time, the data for benzoic acid in narrow pipes agree better with the present author's experimental results, in which the spectrophotometric method was used for measuring the solution rate of benzoic acid and with the values of Gzovsky and Planovsky [2] who studied the solution of the layers of oxalic acid deposited on the internal pipe surface and used the conductometric method. Gzovsky and Planovsky studied mass transfer over a comparatively narrow range of Prandtl

Table 1. *Comparison of experimental Nusselt numbers with those predicted by equation* (23), (*mass transfer*). $Re = 10\,000$

Pr	Experimental *Nu*	Reference	*Nu* predicted by equ. (23)	Note
900	600	6	650	Obtained in experiments with the 1st cylinder
1000	500	4	680	
1000	1030	2	680	
1300	520	4	770	
2150	410	4	970	
2240	1500	4	1000	
3100	590	4	1160	

Table 2. *Comparison of experimental Nusselt number with those predicted by equation* (23), (*heat transfer*). $Re = 10\,000$

Pr	Experimental *Nu*	Reference	*Nu* predicted by equ. (23)	Divergence from *Nu* predicted by equ. (23) (per cent)
37	165	7	179	−7·8
41	170	8	185	−8·1
55	195	9	203	−3·9
58·5	193	7	207	−6·7
93	245	9	245	0
95	245	7	247	−0·8
135–152	330	9	290	+13·7
185	315	9	322	−2·2
298–346	420	9	407	+3·1
340	380	9	415	−8·4
590	480	9	533	−9·9

and Reynolds numbers and did not eliminate the effect of the approach length. They proposed the empirical formula

$$Nu = 0{\cdot}00062\ Re^{1{\cdot}15}\ Pr^{0{\cdot}54} \qquad (24)$$

In their experiments Meyerink and Friedlander [6] solved the internal surface of hollow cylinders made of compressed aspirin, benzoic and cinnamic acids in a turbulent water flow. The rate of solution was found from the change of the cylinder weight as in [4]. Again the data of that work cannot be considered as reliable, since the Nusselt numbers were strongly affected by the length-to-diameter ratio of the test-section, although the test section was preceded by a straight approach length 50 diameters long. In Linton and Sherwood's experiments such a dependence was not observed.

One of the main disadvantages of determining the rate of solution from the weight changes is that in some cases water or another solvent absorbed by the sample during the experiment may be very difficult to remove by drying. Meyerink and Friedlander noticed that samples of benzoic acid absorbed very little water and were dried without difficulties, whereas the samples of cinnamic acid had to be dried for 16–20 h. This might be a possible explanation for the fact that the solution rates obtained with β-naphthol and cinnamic acid were lower than those for benzoic acid in the experiments of Linton and Sherwood.

Table 1 contains several experimental values of Nu at $Re = 10^4$, which are compared with those predicted by equation (23). It is seen that the scatter of the experimental values is so great that they cannot possibly be used for the verification of any theoretical formulae. Accurate experiments on mass transfer from a pipe wall to turbulent fluid flow are therefore badly needed.

The position is better with respect to the heat-transfer data at $Pr \gg 1$. The available published values of heat transfer are sufficiently reliable and agree well [7–9] among themselves. In Table 2 experimental data for the heat transfer at $Pr \gg 1$ are compared with the values of Nu predicted by equation (23). The scatter can be considered to be within the magnitude of the experimental error especially if we note that Friend and Metzner [9] reported that the experimental error in the determination of the thermal conductivity of the viscous materials used by them (molasses, maize sirup) could have been ten per cent.

In conclusion it should be emphasized that agreement of the predicted values with the experimental data on heat transfer at $Re = 10^4$ and $Pr = 37$–590 must not be taken as a confirmation of the physical assumptions made in the derivation of the theoretical equation.

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Abstract—The derivation of a heat- and mass-transfer equation at $Pr \gg 1$ is presented which is based on consideration of transfer in a viscous sublayer alone. The turbulent transfer coefficient in a viscous sublayer is assumed to be proportional to the squared distance from the wall. Predicted values are compared with experimental data.

Résumé—On présente la déduction d'une équation de transport de chaleur et de masse pour $Pr \gg 1$, qui est basée sur la considération du transport uniquement dans une sous-couche visqueuse. On suppose que le coefficient de transport turbulent dans une sous-couche visqueuse est proportionnel au carré de la distance de la paroi. Les valeurs prévues sont comparées avec les données expérimentales.

Zusammenfassung—Es wird die Ableitung einer Wärme-und Stoffübergangsgleichung für $Pr \gg 1$ angegeben, die auf der Betrachtung der Übertragung in einer viskosen Unterschicht allein basiert. Die turbulente Transportgrösse in einer viskosen Unterschicht wird proportional zum Quadrat des Wandabstandes angenommen. Vorherbestimmte Werte werden mit Versuchsdaten verglichen.